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Note

The subdivision graph of a graceful tree is a graceful tree

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Abstract

Koh, Rogers and Tan (Discrete Math. 25 (1979) 141–148) give a method to construct a bigger graceful tree from two graceful trees. Based upon their results, we give a new construction, which allows us to prove that the subdivision graph of a graceful tree is still a graceful tree.

According to [4], let $T(n)$ be a tree on n vertices. A *valuation* on $T(n)$, is a bijection θ from the vertex-set of $T(n)$ onto the set $N=\{1,2,\ldots,n\}$. For each edge uv in $T(n)$, the *weight* of uv, denoted by $\theta(uv)$, is the value $|\theta(u)-\theta(v)|$. The system $(T(n),\theta)$ is said to be *graceful* if the weights of all edges of $T(n)$ are distinct, then are exactly the integers $\{1,2,\ldots,n-1\}$. A tree T is called a *graceful tree* if there exists a valuation θ on T, such that the system (T,θ) is graceful. In this case, θ is called a *graceful valuation* on T.

The Ringel–Kotzig–Rosa conjecture states that all trees are graceful (see [3,5]). Most of the results about this conjecture are based on constructive methods generating graceful trees from smaller ones. In [4], Koh et al. give a method to construct a bigger graceful tree from two given graceful systems, which is based on the following construction.

The Δ-construction. Let $(T(m),\theta_m)$ and $(T(n),\theta_n)$ be two graceful systems, where $T(m)=[w_1,\ldots,w_m\}$ and $T(n)=\{v_1,\ldots,v_n\}$. Consider an arbitrary fixed vertex v^* in $T(n)$. Based upon the tree $T(m)$, adjoin an isomorphic copy $T_i(n)$ of $T(n)$ to each vertex w_i, $i=1,\ldots,m$, in such a way that v^* and w_i are identified. All the m copies of $T(n)$ are pairwise disjoint and no extra edges are added. Such a new tree with mn vertices is denoted by $T(m)\Delta T(n)$.

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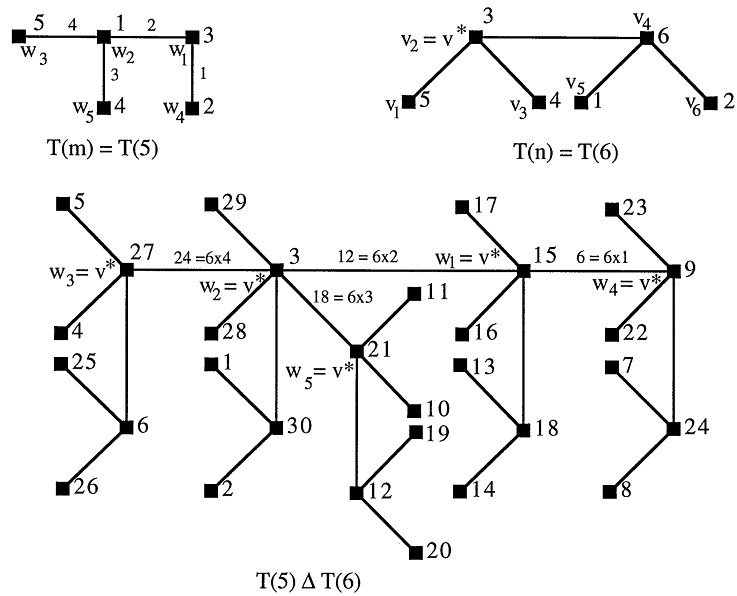


Fig. 1.

Theorem 1 (Koh, Rogers and Tan [4, Theorem 3]). *The mapping* $\theta : T(m)\Delta T(n) \to \{1,\dots,mn\}$ *defined by*

$$\theta(v) = \begin{cases} (\theta_m(w_i) - 1)n + \theta_n(v) & \text{if } d(v^*,v) \text{ is even},\\ (m - \theta_m(w_i))n + \theta_n(v) & \text{if } d(v^*,v) \text{ is odd} \end{cases}$$

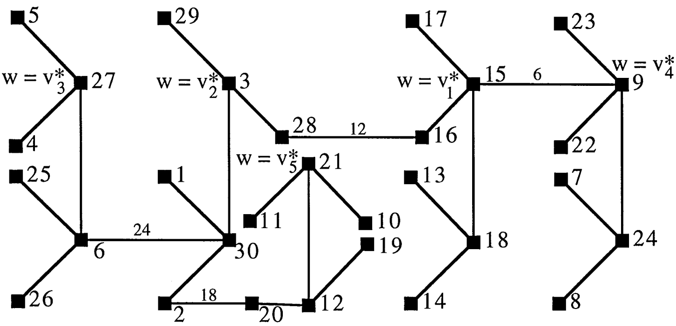
for each v *in* $T_i(n)$, $i = 1,\dots,m$, *where the distance* d *is taken in* $T(n)$, *is a valuation on* $T(m)\Delta T(n)$ *and the system* $(T(m)\Delta T(n), \theta)$ *is graceful.*

In Fig. 1 we reproduce an example of [4], which illustrates the Δ-construction.

The generalized Δ-construction. In the proof of Theorem 1 it is explicitly observed that if w_iw_j is an edge in $T(m)$, $\theta(w_iw_j) = n\theta_m(w_iw_j)$. Moreover, if $v^{(i)}$ is the corresponding vertex in $T_i(n)$ of the vertex v in $T(n)$, we observe that if w_iw_j is an edge in $T(m)$, then $|\theta(v^{(i)} - \theta(v^{(j)})| = \theta(w_iw_j)$ for each vertex v in $T(n)$. Therefore the Δ-construction can be generalized by connecting $T_i(n)$ and $T_j(n)$ with any edge $v^{(i)}v^{(j)}$ instead of the fixed edge w_iw_j. We call this construction a *generalized Δ-construction*. In Fig. 2 we given an example of it.

Remark. For fixed $T(m)$ and $T(n)$, $m \neq 1$, the Δ-construction gives at most n different graceful trees of order mn, one for each choice of v^* in $T(n)$, whereas the generalized Δ-construction gives n^{m-1} $(m \geqslant 1)$ different graceful trees of order mn, for each fixed vertex v^*.

The Δ_{+1}-construction. If p is a prime, the Δ-construction, even generalized, is useless to construct graceful trees of order p using smaller graceful trees. We give a



T(5) Δ T(6) generalized

Fig. 2.

construction, based on the generalized Δ-construction, which can be useful also in such a case.

Set $(T(m),\theta_m)$ and $(T(n),\theta_n)$ as above. Let v^* be an arbitrary fixed vertex in $T(n)$ and w the vertex of $T(m)$ with $\theta_m(w)=m$. Consider $T(m)-w$, which in general is not a connected tree, but just a disjoint union of trees.

We can still perform the generalized Δ-construction as follows.

Let $\overset{<}{v}$ (resp $\overset{>}{v}$) be the vertex of $T(n)$ with $\theta_n(\overset{<}{v}=1$ (resp. $\theta_n(\overset{>}{v})=n$). Remark that $\overset{<}{v}\,\overset{>}{v}$ is an edge of $T(n)$ so $d(v^*,\overset{<}{v})$ and $d(v^*,\overset{>}{v})$ do not have the same parity.

$$\text{Set}\quad (T(n),\tilde{\theta})=\begin{cases}(T(n),\theta_n) & \text{if } d(v^*,\overset{<}{v}) \text{ is even}\\[2ex] (T(n),\theta'_n) & \text{if } d(v^*,\overset{<}{v}) \text{ is odd}\end{cases}$$

where θ'_n is the vertex complement valuation of θ_n defined by $\theta'_n(v)=n+1-\theta_n(v)$, for each v in $T(n)$ see [6]).

Using the Δ-construction, construct the graph $G=(T(m)-w)\Delta T(n)$ with $(m-1)n$ vertices, which in general is a disjoint union of trees, and consider the mapping $\theta: G\to\{1,\ldots,(m-1)n\}$, defined by (see Theorem 1)

$$\theta(v)=\begin{cases}(\theta_m(w_i)-1)n+\tilde{\theta}(v) & \text{if } d(v^*,v) \text{ is even}\\[2ex] (m-1-\theta_m(w_i))n+\tilde{\theta}(v) & \text{if } d(v^*,v) \text{ is odd}\end{cases}$$

for each v in $T_i(n)$, $i=1,\ldots,m-1$.

For each edge vv' in G the values $|\theta(v)-\theta(v')|$ are different and the missing values from $\{1,\ldots,(m-1)n-1\}$ are exactly the multiples np_k, where $p_k=\theta_m(w,w_k)$ are the weights of the edges incident to w in $T(m)$. We can now recover the missing values in the following way. Add to G a new vertex u and, for each k such that ww_k is an edge in $T(m)$, add to G the edges $u\overset{<}{v}{}^{(k)}$, if $\tilde{\theta}=\theta_n$ (resp. $u\overset{>}{v}{}^{(k)}$, if $\tilde{\theta}=\theta'_n$), where $\overset{<}{v}{}^{(k)}$ (resp. $\overset{>}{v}{}^{(k)}$) is the corresponding vertex of $\overset{<}{v}$ (resp. $\overset{>}{v}$) in $T_k(n)$. The graph obtained in a tree and it is denoted by $T(m)\Delta_{+1}T(n)$. Moreover,

Theorem 2. *The mapping* $\theta_{+1}: T(m)\Delta_{+1}T(n) \to \{1,\dots,(m-1)n+1\}$ *defined by*

$$\theta_{+1}(v)=\theta(v) \quad \textit{for each } v \textit{ in } G \qquad \textit{and} \quad \theta_{+1}(u)=(m-1)n+1$$

is a valuation on $T(m)\Delta_{+1}T(n)$ *and the system* $(T(m)\Delta_{+1}T(n),\ \theta_{+1})$ *is graceful.*

Proof. If $\tilde{\theta}=\theta_n$, it is sufficient to check that $\theta_{+1}(u\overset{<}{v}^{(k)})=np_k$. But

$$\theta_{+1}(u\overset{<}{v}^{(k)})=|\theta_{+1}(u)-\theta_{+1}(\overset{<}{v}^{(k)})|=|(m-1)n+1-[(\theta_m(w_k)-1)n+1]|$$

$$=|(m-1)n+1-[(m-p_k-1)n+1]|=|p_k n|=p_k n\,.$$

The case $\tilde{\theta}=\theta'_n$ is similar. □

Figs. 3(a) and 3(b) give two examples of Δ_{+1}-constructions.

Observe that in Fig. 3(a) the Δ_{+1}-construction gives a graceful tree of order 5, obtained non-trivially i.e. not connecting just 5 to 1, from a graceful tree of order 4.

Since in Fig. 3(b) $d(v^*,\overset{<}{v})$ is odd we use the vertex complement valuation on $T(7)$ to obtain a graceful valuation on $T(5)\Delta_{+1}T(7)$, which connects u and the different copies $T_i(7)$ according to the adjacences in $T(5)$.

Remark. The Δ-construction is well-behaving with respect to vertex complement valuation, i.e. if θ'_m (resp. θ'_n) is the vertex complement valuation on $T(m)$ (resp. $T(n)$) of θ_m (resp. θ_n) it follows that the vertex complement valuation θ' of θ on $T(m)\Delta T(n)$ can be defined as in Theorem 1 fixing the same vertex v^* in $T(n)$ and using θ'_m and θ'_n instead of θ_m and θ_n.

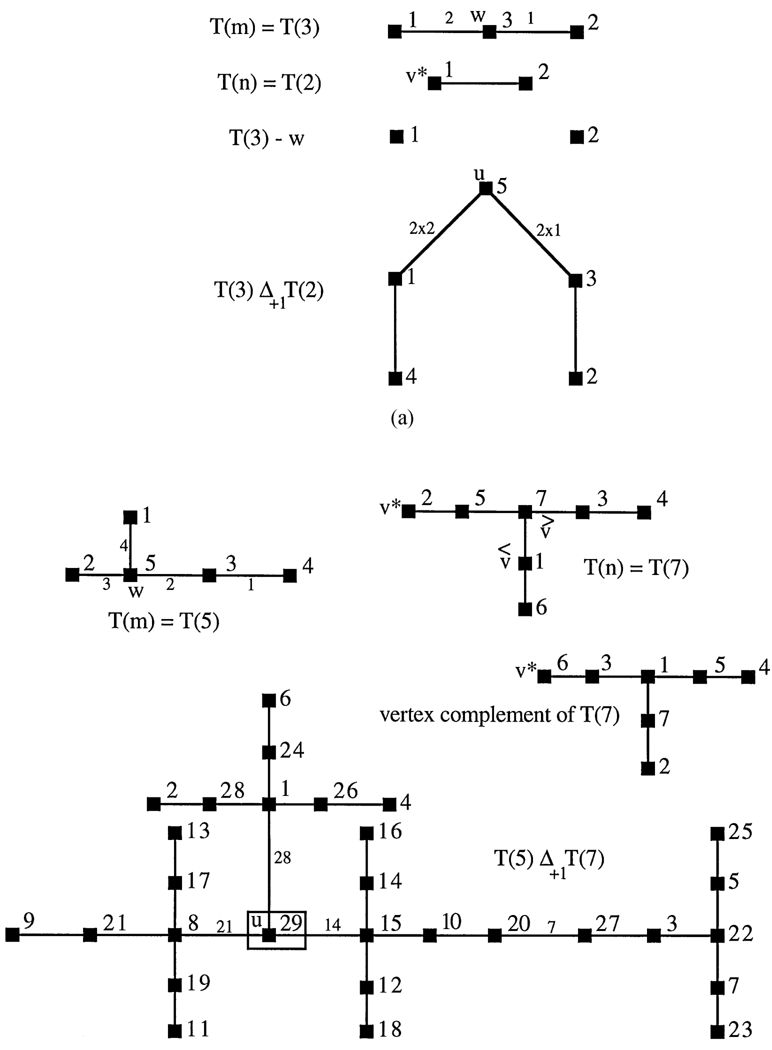
We want to sketch a Δ_{+1}-construction which gives the vertex complement valuation of θ_{+1}. Label each vertex v in $T(n)$ by

$$\tilde{\theta}(v)=\begin{cases}\theta'_n(v)+1 & \text{if } d(v^*,\overset{<}{v}) \text{ is even,}\\ n+2-\theta'_n(v) & \text{if } d(v^*,\overset{<}{v}) \text{ is odd.}\end{cases}$$

Construct $G=(T(m)-w)\Delta T(n)$ and let $\theta: G\to\{2,\dots,(m-1)n\}$ be defined by

$$\theta(v)=\begin{cases}(\theta_m(w_i)-1)n+\tilde{\theta}(v) & \text{if } d(v^*,v) \text{ is odd,}\\ (m-1-\theta_m(w_i))n+\tilde{\theta}(v) & \text{if } d(v^*,v) \text{ is even.}\end{cases}$$

Add to G a new vertex z labelled 1 and, for each k such that ww_k is an edge in $T(m)$, add the edge $z\overset{<}{v}^{(k)}$ if $d(v^*,\overset{<}{v})$ is even (resp. $z\overset{>}{v}^{(k)}$ if $d(v^*,\overset{<}{v})$ is odd). In this way we obtain a tree and the labelling on its vertices is the vertex complement valuation of θ_{+1}.



(b)

Fig. 3.

Remark. Theorem 1 of [4] can be obtained as a corollary of Theorem 2 by choosing as $T(m)$ the star $K_{1,p}$ with center w.

If each edge $e = uv$ of a graph G is replaced by a new vertex w and the edges uw and vw, then the resulting graph is called the *subdivision graph of* G and is denoted by $S(G)$.

Theorem 3. *The subdivision graph of a graceful tree is a graceful tree.*

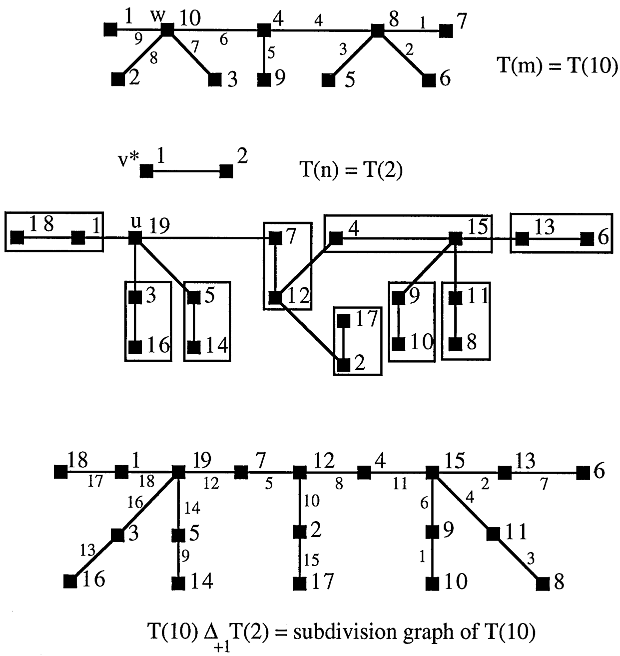


Fig. 4.

Proof. Let $(T(m), \theta_m)$ be a graceful system, w the vertex of $T(m)$ with $\theta_m(w) = m$, and $(T(2), \theta_2)$ be the graceful system with vertices $\overset{<}{v}$ and $\overset{>}{v}$ such that $\theta(\overset{<}{v}) = 1$ and $\theta(\overset{>}{v}) = 2$. Fix in $T(2)$ the vertex $v^* = \overset{<}{v}$ and perform the Δ_{+1}-construction obtaining $T(m)\Delta_{+1}T(2)$. Using the generalized Δ-construction, connect the copies $T_i(2)$ and $T_j(2)$ in $T_m - w$ by the edge $\overset{<}{v}^{(i)}\overset{<}{v}^{(j)}$ (resp. $\overset{>}{v}^{(i)}\overset{>}{v}^{(j)}$ if $d(w_j, w) = d(w_i, w) + 1$ is odd (resp. even). Then $T(m)\Delta_{+1}T(2) = S(T_m)$ and the system $(S(T_m), \theta_{+1})$ is graceful by Theorem 2. □

In Fig. 4 a graceful system for the subdivision graph of a caterpillar is shown.

Remark. Theorem 3 is not true for graphs. In fact a cycle C_n is graceful if and only if $n \equiv 0$ or 3 (mod 4) (see [5] and also [1]). Therefore C_3 is a graceful graph but its subdivision graph C_6 cannot be graceful.

Proposition 4. *Let $S_n(T)$ be the nth subdivision graph of a tree T, i.e. the tree obtained by inserting n new vertices into each edge of T. Then if T is graceful, $S_n(T)$ is also a graceful tree.*

Proof. It is similar to the proof of Theorem 3 using as $T(n)$ a path of length n, where $\overset{<}{v}$ is an end-vertex. □

Acknowledgements

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